International Journal of Theoretical Physics, Vol. 26, No. 1, 1987

Nordström's Theory in the Light of the Dualistic Gravitation Theory

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Received October 24, 1986

It is shown that the recent dualistic theory of gravitation of the author can be regarded as equivalent to the Nordström theory (1913) supplemented by additional field variables. Such a point of view not only removes certain theoretical shortcomings of the Nordström theory, but also clarifies the relationship between this theory and the Einstein theory.

1. INTRODUCTION

The first theory of gravitation consistent with (1) the conservation of momentum and energy, (2) equality of the gravitational and inertial mass of an isolated material system, and (3) the restricted principle of relativity was the scalar theory of gravitation by Nordström (1913), which can be regarded as the forerunner of Einstein's tensor theory of gravitation (1916). In fact, Einstein was for some time very much taken with this theory and published papers (Einstein, 1913, 1914) on it prior to the advent of his general theory in 1915. Einstein and Fokker (1914) first gave a fully covariant formulation of Nordström's theory which even today may be regarded as the best presentation of it.

One reason why scientific opinion ultimately turned away from the Nordström theory in spite of its simplicity and elegance was the discovery in 1919 that light rays are deflected by a strong gravitational field. According to the Nordström theory, light rays should be unaffected by a scalar gravitational field, as a consequence of the fact that null geodesics are given by the same equations in a flat space-time and a conformally flat space-time, which is used in the Nordström theory. In addition, the perihelion rotation is also obtained incorrectly in this theory.

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0020-7748/87/0100-0063$05.00/0

Thus, the Nordström theory has since been only of academic interest, although even in 1953 Gürsey (1953) pointed out that it could serve as a viable theory for cosmological models. This was recently confirmed in detail in a thesis (Mohammad, 1980) under the author's supervision, where it was shown that the Nordström theory even restricts the possibilities of cosmological models to two only, namely spatially flat ($k=0$) and open universes ($k=-1$), the closed universe ($k=+1$) being ruled out.

Still more recently, the author Mahanta (1984) and Villi (1985) have advocated the use of conformally flat space-time (which forms the background of the Nordström theory) for microscopic physics. Of course the motivations in the two cases are somewhat different. The second in fact uses the Einstein field equations, the solution of which is supposed to result in a conformally flat space-time, as happens, for instance, in the case of the Friedmann universe or in many other cases of perfect fluid solutions, where indeed the conformally flat space-time results from the Einstein field equations. But in Villi's case it is an assumption only.

Coming to the author's recent work, which has been called "a dualistic approach to gravitation" because it deals with both the microscopic and macroscopic aspects of gravitation in a single framework, it is based on a variational formulation in which, in addition to the metric tensor g_{ij}, another tensor P^{ijkl} with the symmetry properties of R^{ijkl} plays an important role. But the equation determining the scalar function H for the conformally flat space-time is again the Nordström (1913) equation. This suggests that possibility that the new approach is nothing but the Nordström theory with certain additional field variables P^{ijkl}, which are needed to remove theoretical shortcomings in the same, such as the absence of a variational principle.

In this paper we shall address ourselves to a detailed clarification of this question and in passing make a few more points regarding the dualistic approach.

2. IS NORDSTRÖM'S THEORY COMPLETE?

The best way of looking at the Einstein and Nordström theories from a common vantage point is to consider the following irreducible decomposition of the Riemann tensor (Schild, 1962):

$$R^{ij}_{kl} \equiv C^{ij}_{kl} + B^{ij}_{kl} + A^{ij}_{kl} \tag{1}$$

Here R^{ij}_{kl} is the Riemann tensor and C^{ij}_{kl} is the Weyl conformal tensor

$$C^{ij}_{kl} \equiv R^{ij}_{kl} - \tfrac{1}{2}(\delta^j_k Q^i_l + \delta^i_l Q^j_k - \delta^i_k Q^j_l - \delta^j_l Q^i_k) - \tfrac{1}{12} R(\delta^j_k \delta^i_l - \delta^i_k \delta^j_l) \tag{2}$$

$$Q^i_j \equiv R^i_j - \tfrac{1}{4}\delta^i_j R \tag{3}$$

$$B^{ij}_{kl} \equiv \tfrac{1}{2}(\delta^j_k Q^i_l + \delta^i_l Q^j_k - \delta^i_k Q^j_l - \delta^j_l Q^i_k) \tag{4}$$

$$A^{ij}_{kl} \equiv \tfrac{1}{12} R(\delta^j_k \delta^i_l - \delta^i_k \delta^j_l) \tag{5}$$

It is well known that B^{ij}_{kl} is essentially equivalent to the traceless tensor Q^i_j and A^{ij}_{kl} to the scalar R.

The Einstein theory can be derived by assigning

$$Q^j_i \equiv R^j_i - \tfrac{1}{4}\delta^j_i R = k(T^j_i - \tfrac{1}{4}\delta^j_i T) \tag{6}$$

where T^j_i is the energy-momentum tensor satisfying

$$T^j_{i;j} = 0 \tag{7}$$

and also

$$R = -kT \tag{8}$$

Equation (8) is implied by the consistency requirement for equation (6). For, if we write it as

$$R^j_i - \tfrac{1}{2}\delta^j_i R = kT^j_i - \tfrac{1}{4}\delta^j_i(kT + R) \tag{9}$$

then, taking the divergence of both sides, it follows that

$$(kT + R)_{,i} = 0$$

from which $R = -kT$, ignoring the cosmological constant. Equations (6) and (8) are together equivalent to the Einstein field equations

$$R^j_i - \tfrac{1}{2}\delta^j_i R = kT^j_i \tag{10}$$

On the other hand, for the Nordström theory, we put

$$C^{ij}_{kl} = 0 \tag{11}$$

and

$$R = \text{const} \times T \tag{12}$$

$$= (12K/a)T, \quad \text{say} \tag{13}$$

(the reason for taking the constant in this form will be clear when we establish the link with the dualistic approach in the next section).

Condition (11) implies a conformally flat space-time whose metric we shall take in the form

$$ds^2 = H^2 \eta_{ij}\, dx^i\, dx^j \equiv H^2[(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2] \tag{14}$$

H being a scalar function of the x^i and η_{ij} is the Minkowski tensor. Then equation (13) is equivalent to

$$R \equiv (6/H^3)\,\Box H = (12K/a)T \tag{15}$$

Equation (15) is the well-known Nordström equation and

$$\Box H \equiv \frac{\partial^2 H}{(\partial x^0)^2} - \frac{\partial^2 H}{(\partial x^1)^2} - \frac{\partial^2 H}{(\partial x^2)^2} - \frac{\partial^2 H}{(\partial x^3)^2} \tag{16}$$

Thus, to derive at the Einstein theory, we leave out C^{ij}_{kl} in the decomposition (1) from the field equations and to arrive at the Nordström theory we do the same for B^{ij}_{kl} or, equivalently, Q^i_j. In both cases the agency responsible for the gravitational field is T^i_j or something derived from it, such as $T \equiv T^i_i$. This is essential for the validity of the principle of equality of inertial and gravitational masses of isolated systems.

Now it is well known that the Einstein field equations (10) are also derivable from a variational principle. This was shown by Hilbert (1915), who arrived at equations (10) at almost the same time as Einstein (1915). A consequence of this is that the conservation equations

$$T^j_{i;j} = 0 \tag{17}$$

follow from the field equations themselves, as can be seen by taking the divergence of equation (10). This is considered to be a very attractive feature of the Einstein theory. But to date nobody has derived the Nordström field equations (11) and (12) from a variational principle and as a result the conservation equations (17) have to be independently postulated. But once this is done we can easily see by taking the partial derivative of equation (13) with respect to x^i and using (17) and the identities

$$(R^j_i - \tfrac{1}{2}\delta^j_i R)_{;j} = 0 \tag{18}$$

that

$$(R^j_i - \tfrac{1}{4}\delta^j_i R)_{;j} = -(12K/a)(T^j_i - \tfrac{1}{4}\delta^j_i T)_{;j} \tag{19}$$

(see also Appendix).

Equation (19) suggests that even in the Nordström theory there could be some kind of relationship between R^j_i and T^j_i as in the general theory of Einstein, even though the tensors B^{ij}_{kl} and A^{ij}_{kl} in (1) are algebraically independent and we left B^{ij}_{kl} out of the reckoning altogether in deriving the Nordström system (11), (12). If we set

$$R^j_i - \tfrac{1}{4}\delta^j_i R = -(12K/a)(T^j_i - \tfrac{1}{4}\delta^j_i R) \tag{20}$$

then this together with (13), namely $R = (12K/a)T$, will reduce to the Einstein system. This may be possible for a certain class of T^j_i, but not in general, since equations (13) and (20) together represent ten equations (less four identities) for the determination of a single scalar function H. Thus the system is overdetermined in general.

It is clear therefore that if we look for a relationship in the Nordström-type theory between R^j_i and T^j_i, then in general there must be other unknowns present in it and for special values of these unknowns the system must reduce to the Einstein field equations in the form

$$R^j_i - \tfrac{1}{2}\delta^j_i R = -(12K/a)T^j_i$$

It is also desirable that the new variables should play a role in rendering a variational formulation of the Nordström theory possible. This program has been fully implemented in what has been called "a dualistic approach to gravitation" (Mahanta, 1984). In the next section we shall present a brief resume of the formalism from this new point of view.

3. THE DUALISTIC APPROACH AS A SUPPLEMENTED NORDSTRÖM THEORY

The "ansatz" of the dualistic approach is the following variational principle:

$$\delta \int I(-g)^{1/2}\, d^4x = 0 \tag{21}$$

with

$$I \equiv P^{ijkl}\{R_{ijkl} - (g_{jk}d_{il} + g_{il}d_{jk} - g_{ik}d_{jl} - g_{jl}d_{ik})\} + KL + ag^{ij}d_{ij} \tag{22}$$

where g_{ij} is the metric tensor, R_{ijkl} is the curvature tensor, d_{ij} is another symmetric covariant tensor, P^{ijkl} is a fourth-rank tensor with the symmetry properties of R^{ijkl} with 20 algebraically independent components, L is the Lagrangian density of nongravitational fields, and a and K are constants.

Variations of P^{ijkl}, d_{ij}, and g_{ij} lead, respectively, to the following system of equations:

$$R_{ijkl} = g_{jk}d_{il} + g_{il}d_{jk} - g_{ik}d_{jl} - g_{jl}d_{ik} \tag{23}$$

$$P^{ij} \equiv g_{kl}P^{iklj} = (a/4)g^{ij} \tag{24}$$

$$P^{ijkl}_{;il} - \Pi^{jk} - \tfrac{1}{4}a(d^{jk} - dg^{jk}) = KT^{jk} \tag{25}$$

the constant K being redefined and

$$\Pi^{jk} \equiv d_{il}P^{ijkl}, \qquad d \equiv g_{ij}d^{ij} \tag{26}$$

Equation (23) shows that we have a conformally flat space-time (Eisenhart, 1966) and

$$d^{jk} - dg^{jk} = \tfrac{1}{2}(R^{jk} - \tfrac{1}{2}g^{jk}R) \tag{27}$$

and

$$d = R/6 \tag{28}$$

The identities are

$$(P^{ijkl}_{;il} - \Pi^{jk})_{;k} = 0 \tag{29}$$

$$(d^{jk} - dg^{jk})_{;k} = \tfrac{1}{2}(R^{jk} - \tfrac{1}{2}g^{jk}R)_{;k} = 0 \tag{30}$$

in the derivation of which (23) is used. Taking the trace of equation (25), we get

$$\tfrac{1}{2}ad = KT \tag{31}$$

i.e.,

$$R = (12K/a)T \tag{32}$$

Thus, starting from a variational principle (21), we have been able to derive both sets of Nordström's equations, namely (23) and (32), as a consequence of our field equations. We shall now derive (19) directly from the field equations (25) for P^{ijkl} and show how for a specific choice of the P's the Einstein field equations result. For this we first write (25) in the form

$$R^j_k - \tfrac{1}{2}\delta^j_k R = -(8K/a)\{T^j_k + (1/K)(-P^{ij\cdot l}_{k;il} + \Pi^j_k)\} \tag{33}$$

which can also be reduced to

$$(R^j_k - \tfrac{1}{4}\delta^j_k R) = -\frac{8K}{a}\left[T^j_k - \frac{1}{4}\delta^j_k T + \frac{1}{K}(-P^{ij\cdot l}_{k;il} + \Pi^j_k)\right] + \frac{1}{12}R\delta^j_k \tag{34}$$

using equation (32).

Taking the divergence of both sides and using (29) and

$$(R^j_k - \tfrac{1}{4}\delta^j_k R)_{;j} = \tfrac{1}{4}R_{,k}$$

we get

$$(R^j_k - \tfrac{1}{4}\delta^j_k R)_{;j} = -(12K/a)(T^j_k - \tfrac{1}{4}\delta^j_k T)_{;j} \tag{35}$$

The system will reduce to the Einstein system if we put in equation (33)

$$P^{ijkl} = (a/12)(g^{jk}g^{il} - g^{ik}g^{jl}) \tag{36}$$

A contraction shows $P^{jk} = (a/4)g^{jk}$ as required and

$$\begin{aligned}\Pi^{jk} = d_{il}P^{ijkl} &= -(a/12)(d^{jk} - dg^{jk})\\ &= -(a/24)(R^{jk} - \tfrac{1}{2}Rg^{jk})\end{aligned} \tag{37}$$

Thus equation (33) reduces to

$$R^j_k - \tfrac{1}{2}\delta^j_k R = -(8K/a)T^j_k - \tfrac{1}{3}(R^j_k - \tfrac{1}{2}\delta^j_k R)$$

i.e.,

$$R^j_k - \tfrac{1}{2}\delta^j_k R = -(12K/a)T^j_k \tag{38}$$

the Einstein field equations (for a conformally flat space-time). In other words, the condition that the Nordström theory is equivalent to the Einstein case is precisely given by equation (36).

Actually what occurs is the following. In the dualistic approach the conformally flat space-time is in a sense imposed on the microscopic world for any arbitrary T_i^j. Mathematically this can be achieved with the help of an extra gauge field P^{ijkl} whose field equations turn out to be equation (25). Now it may happen that for a certain class of T_i^j the conformally flat space-time results from the Einstein field equations themselves (as in the case of many perfect fluid solutions in general relativity). In such a case the P-field is redundant and equation (36) is the condition for this. Rewriting (36) in the form

$$C^{ijkl} \equiv P^{ijkl} - (a/12)(g^{jk}g^{il} - g^{ik}g^{jl}) = 0 \tag{39}$$

We may thus interpret C^{ijkl} as a measure of the deviation from the representation of a system by the Einstein field equations for a conformally flat space-time. This was the consideration that led the author to identify the above tensor in the averaged form with the conformal tensor of Weyl for the macroscopic space-time resulting from averaging equations (33) over hadronic space-times (Mahanta, 1984).[2]

As regards the role of the P-field in microscopic physics, obviously it is related to hadronic processes, but only a detailed investigation can reveal its precise physical meaning. But it is amply clear from the above discussion that the P-field fills a big gap in the theoretical structure of the Nordström theory and seems to be the agent that generates the conformal part of the macroscopic curvature tensor.

APPENDIX

The relation (19) is of course only one of an entire class of similar relationships. To get this class we proceed as follows:

We may write generally

$$R_{,i} = -(1/\lambda)\{R_i^j - (\tfrac{1}{2} + \lambda)\delta_i^j R\}_{;j} \tag{A1}$$

$$T_{,i} = -(1/\mu)(T_i^j - \mu\delta_i^j T)_{;j} \tag{A2}$$

Then, from $R_{,i} = (12K/a)T_{,i}$ we get

$$\{R_i^j - (\tfrac{1}{2} + \lambda)\delta_i^j R\}_{;j} = (12K/a)(\lambda/\mu)(T_i^j - \mu\delta_i^j T)_{;j} \tag{A3}$$

of which relation (19) is a special case with $\lambda = -\mu = -1/4$.

[2]The guiding idea in this is that although in the general case the system (23)-(26) differs from the Einstein system, when averaged over hadronic space-times it results in the Einstein equations of general relativity. This is the transition from the microscopic system to the macroscopic system.

To derive (A3) directly from (33), we rewrite the latter as

$$R^j_i-(\tfrac{1}{2}+\lambda)\delta^j_i R$$
$$=-\frac{8K}{a}\left[(T^j_k-\mu\delta^j_k T)+\frac{1}{K}(-P^{ij\cdot l}_{k;il}+\Pi^j_k)\right]-\lambda\delta^j_k R-\frac{8K}{a}\mu\delta^j_k T$$

Taking the divergence of both sides and using (29), (A1), (A2), and (32), we immediately get (A3). To arrive at the Einstein field equations, we first put

$$R^j_i-(\tfrac{1}{2}+\lambda)\delta^j_i R=(12K/a)(\lambda/\mu)(T^j_i-\mu\delta^j_i T) \tag{A4}$$

The trace equation will agree with the Nordström equation if we take $\lambda=-\mu$ and then equation (A4) reduces to the Einstein equation.

ACKNOWLEDGMENTS

The author would like to express his acknowledgment and thanks to the Indian Institute of Advanced Study for the grant of a Fellowship and the facilities for this work.

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